A

COLLECTION

OF

DIOPHANTINE PROBLEMS

WITH

SOLUTIONS.

COMPILED BY

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WASHINGTON:
ARTEMAS MARTIN.
1888.

INTRODUCTORY NOTE.

THE following pages were left in sheets by the late Dr. MATTESON at his death, which occurred on Dec. 15, 1876. Had he lived, he would undoubtedly have added many pages of interesting problems and solutions to the existing number.

Having recently purchased from the guardian of the heirs the entire edition, consisting of only a few hundred copies, I decided to issue it with the addition of cover, title-page, and this note.

The authorship of the solutions, as near as can now be ascertained, is given below. A considerable portion of this information has been furnished by Mr. REUBEN DAVIS, Sr., of Clifton, Kansas, one of the most ingenious of living "Diophantists," who rendered valuable assistance to Dr. MATTESON in the compilation of the work.

The solutions of Problems 1 and 4 are by the late ABIJAH MC LEAN, Esq., of New Lisbon, Ohio, who was very skillful in handling Diophantine Problems of great difficulty.

The solutions of Problems 5, 7, 9, 22 and 24 (except the last two paragraphs of the solution of 22 which are by Mr. DAVIS) are from the pen of the late lamented Dr. DAVID SHERMAN HART, of Stonington, Conn., who was one of the ablest, if not *the* ablest, of the Diophantine writers of his time.

Dr. HART also rendered efficient aid to the compiler in the revision of many difficult solutions.

The solutions of Problems 6, 8 and 12 (except the last two paragraphs of the solution of 12 which are by Mr. DAVIS) are by Dr. MATTESON.

The other solutions are mostly the work of Mr. DAVIS, with some slight modifications by Dr. MATTESON.

The following errors have been noted by Dr. HART, Mr. DAVIS, and the writer:

Page 4, line 3 from the bottom, *for* "because y is *odd*" *read* because x is *odd*.

Page 7, line 23 from bottom, *for* "three squares sought" *read* three equi-different numbers from which are obtained the three squares sought.

Page 8, line 1, *for* "of the natural series" *read* in the natural series.

Page 8, line 2 from bottom, *for* "nad" *read* and.

Page 9, where "of the natural series" occurs, *read* in the natural series.

Page 11, line 13, *for* "$13 = 3^2 + 1^2$" *read* $13 = 3^2 + 2^2$.

Page 14, formula (14) should be

$$x = \frac{[(-ab+ac+bc)^2 - 4abc^2]^2}{8abc(-ab+ac+bc)(ab-ac+bc)(ab+ac-bc)},$$

and to preserve the sign of x unchanged in (14) as well as in (15), the relations between a, b, c, must be such that the sum of any two exceeds the other.

Page 15, line 3, right-hand member of equation, *for* "$-b^2c^2$" *read* $-3b^2c^2$.

ARTEMAS MARTIN.

Washington, D. C., July 2, 1888.

Diophantine Problem.

It is required to find four affirmative integer numbers, such that the sum of every two of them shall be a cube.

Solution.

If we assume the first $=\frac{1}{2}(x^3+y^3-z^3)$, the second $=\frac{1}{2}(x^3-y^3+z^3)$, the third $=\frac{1}{2}(-x^3+y^3+z^3)$, and the fourth $=v^3-\frac{1}{2}(x^3+y^3-z^3)$; then, the first added to the second $=x^3$, the first added to the third $=y^3$, the second added to third $=z^3$, and the first added to the fourth $=v^3$.

Thus four of the six required conditions are satisfied in the notation. It remains, then, to make the second plus the fourth $=v^3-y^3+z^3=$ cube, say $=w^3$, and the third plus the fourth $=v^3-x^3+z^3=$ cube, say $=u^3$. Transposing, we have to resolve the equalities $v^3+z^3=w^3+y^3=u^3+x^3$; and with values of x, y, z, in such ratio, that each two shall be greater than the third.

Let us first resolve, in general terms, the equality $v^3+z^3=w^3+y^3$. Taking $v=a+b$, $z=a-b$, $w=c+d$, $y=c-d$, the equation, after dividing by 2, becomes $a(a^2+3b^2)=c(c^2+3d^2)$. Now assume $a=3np+3mq$, $b=mp-3nq$, $c=3nr+3ms$, and $d=mr-3ns$. Substituting these in the preceding equation, it becomes $(3np+3mq)[(3np+3mq)^2+3(mp-3nq)^2]=(3nr+3ms)\times[(3nr+3ms)^2+3(mr-3ns)^2]$; or $(3np+3mq)(p^2+3q^2)3(m^2+3n^2)=(3nr+3ms)(r^2+3s^2)\times 3(m^2+3n^2)$; which, dividing by the common factors $3.3(m^2+3n^2)$, reduces to $(np+mq)(p^2+3q^2)=(nr+ms)(r^2+3s^2)$; or, $np(p^2+3q^2)+mq(p^2+3q^2)=nr(r^2+3s^2)+ms(r^2+3s^2)$;

$\therefore\ m:n::r(r^2+3s^2)-p(p^2+3q^2):q(p^2+3q^2)-s(r^2+3s^2)$; and, if we take $m=r(r^2+3s^2)-p(p^2+3q^2)$, then $n=q(p^2+3q^2)-s(r^2+3s^2)$.

$$\therefore\quad a=3np+3mq=(3rq-3ps)(r^2+3s^2),$$
$$b=mp-3nq=(pr+3qs)(r^2+3s^2)-(p^2+3q^2)^2,$$
$$c=3nr+3ms=(3rq-3ps)(p^2+3q^2),\ \text{and}$$
$$d=mr-3ns=(r^2+3s^2)^2-(pr+3qs)(p^2+3q^2),\ \text{and, at once}$$
$$v=a+b=(3rq-3ps+pr+3qs)(r^2+3s^2)-(p^2+3q^2)^2,$$
$$z=a-b=(3rq-3ps-pr-3qs)(r^2+3s^2)+(p^2+3q^2)^2,$$
$$w=c+d=(3rq-3ps-pr-3qs)(p^2+3q^2)+(r^2+3s^2)^2,$$
$$y=c-d=(3rq-3ps+pr+3qs)(p^2+3q^2)-(r^2+3s^2)^2.$$

We have thus arrived at general expressions for the values of v, z, w, y, such that $v^3+z^3=w^3+y^3=18(rq-ps)(r^2+3s^2)(p^2+3q^2)\times[(r^2+3s^2)^3-(2pr+6qs)(r^2+3s^2)(p^2+3q^2)+(p^2+3q^2)^3]$, as will appear from actual involution, &c.

But, to have the final answer in positive numbers, v must nearly $=3z$, and w nearly $=3y$.

Taking $p=6$, $q=14$, $r=7$, and $s=14$; there results

$$v=48.49.14.13-48^2.13^2=48.49.14-48^2.13=13.2976,$$
$$z=-49.49.12.13+48^2.13^2=-49.49.12+48^2.13=13.1140,$$
$$w=-49.48.12.13+49^2.13^2=-49.48.12+49^2.13=13.2989,$$
$$y=48.48.14.13-49^2.13^2=48.48.14-49^2.13=13.1043.$$

Having now, (dividing by the common factor 13),

$(2976)^3+(1140)^3=(2989)^3+(1043)^3=27838714176=7^3.3^3.4^3.13.3613$;

it remains to find two other cubes, u^3 and x^3, such that $u^3+x^3=7^3.3^3.4^3.13.3613$. This we shall accomplish, if we succeed in dividing $13.3613=(1^2+32^2).(55^2+3.14^2)$ into two cubes. Now, it is demonstrable that when the sum of two cubes is of the form $(a^2+3b^2)(c^2+3d^2)$; and the relation of the quantities is such that $(a^2+3b^2)=$ either $3d-c$, or $c-3d$, the cubes themselves will be $\frac{1}{8}(a^2+3b^2+c+d)^3$ and $\frac{1}{8}(a^2+3b^2-c-d)^3$, for the sum of these cubes equated to the assumed sum, is

$$\tfrac{1}{8}[2(a^2+3b^2)^3+6(c+d)^2(a^2+3b^2)]=(a^2+3b^2)(c^2+3d^2),$$

$$\text{or}\quad (a^2+3b^2)^2+3(c+d)^2=4(c^2+3d^2),$$

$$\text{or}\quad (a^2+3b^2)^2=c^2-6cd+9d^2=(c-3d)^2=(3d-c)^2.$$

Wherefore, $a^2+3b^2=c-3d$, or, $=3d-c$, as was to be shown.

In the example before us, $a=1$, $b=2$, $c=55$, $d=14$, and $a^2+3b^2=13=c-3d=55-3.14$; then $\frac{1}{2}(a^2+3b^2+c+d)=41$, and $\frac{1}{2}(a^2+3b^2-c-d)=-28$, and $(41)^3-(28)^3=13.3613$; and, instead of the *sum*, we have found the *difference* of two cubes$=13.3613$.

But, Mathematicians have shown that

$$a^3-b^3=\left(\frac{a(a^3-2b^3)}{a^3+b^3}\right)^3+\left(\frac{b(2a^3-b^3)}{a^3+b^3}\right)^3;$$ in which, taking $a=41$, $b=28$, we immediately obtain

$$\left(\frac{1081640}{30291}\right)^3+\left(\frac{341899}{30291}\right)^3=13.3613.$$

Multiplying, now, by the reserved cube factors, $7^3.3^3.4^3$, we find $u=\frac{30285920}{10097}$, and $x=\frac{9573172}{10097}$, values that likewise have the proper ratio to insure positive numbers in the question under solution.

Thus, for $v^3+z^3=w^3+y^3=u^3+x^3$, we have found

$$\left(\frac{30285920}{10097}\right)^3+\left(\frac{9573172}{10097}\right)^3=(2989)^3+(1043)^3=(2976)^3+(1140)^3,$$

or, since y is odd, multiply by 10097×2, for integers, and we have

$$v=60571840,\ x=23021160,\ y=21062342,\ z=19146344.$$

With these values, we at once obtain the following answer:

$$\tfrac{1}{2}(x^3+y^3-z^3)=2080913082956455142636,$$

$$\tfrac{1}{2}(x^3-y^3+z^3)=4937801347510680732948,$$

$$\tfrac{1}{2}(-x^3+y^3+z^3)=7262810476410016163052,$$

$$v^3-\tfrac{1}{2}(x^3+y^3-z^3)=214972108693241589340948.$$

Being four numbers as required, such, that the sum of every two of them is a cube number; resulting in the six cubes, $(19146344)^3$, $(21062342)^3$, $(23021160)^3$, $(60097344)^3$, $(60359866)^3$, and $(60571840)^3$.

This answer, it will be observed, is the same as that contained in the *Mathematical Miscellany*, which was communicated by Wm. Lenhart, where it was found from an inspection of a Table of Cube Numbers. Here, the given solution is independent of Tables, and we believe the numbers found to be the least possible.

OBSERVATION: With regard to the assumed numeral values of p, q, r, s, in this solution, we began with the smallest integers, 0, inclusive, and soon found the numbers 6, 14, 7, 14, to answer the purpose in hand, namely: to produce four cubes such, that $v^3+z^3=w^3+y^3$, and which shall have the proper ratio to produce positive numbers in the answer. There are, however, an indefinite number of values for p, q, r, s, that will result in precisely the same values of v, z, w, y, when reduced to the lowest terms.

In fact, any values of p, q, r, s, of the modulus $p=mp'+3nq'$; $q=-np'+mq'$; $r=mr'+3ms'$; $s=-nr'+ms'$; will, after dividing, result in the same values of v, z, w, y, since m and n thus disappear, and the general expressions for v, z, w, y, remain of precisely the former modulus. Thus, if p', q', r', s', be respectively 6, 14, 7, 14, and $m=1$, $n=2$; then $p=90$, $q=2$, $r=91$, $s=0$, and the same answer results, after dividing by $(m^2+3n^2)^2=169$. If $p'=6$, $q'=14$, $r'=7$, $s'=14$, $m=14$, $n=9$; then $p=462$, $q=142$, $r=476$, $s=133$; and now, if $m'=55$, $n'=14$, then $p=31374$, $q=1342$, $r=31766$, $s=651$; all which will result in the same answer.

We have given a successful method of finding two cubes, whose *sum*=13.3613; or rather, in the first instance whose *difference*=13.3613; and here we desire to add, that when such cubes are integral, they may be found by assuming $f+g$, and $f-g$ for the roots, when the sum is an even number; or $\frac{1}{2}(f+g)$ and $\frac{1}{2}(f-g)$, when the sum is odd In this case it is odd; then $\frac{1}{2}(f+g)^3+\frac{1}{2}(f-g)^3=13.3613$; consequently, $f(f^2+3g^2)=4.13.3613$.

Now, since 3613 is a prime number, evidently greater than f, f, if an integer, must be one of the factors, 1, 2, 4, 13, 26, or 52. On trial, we succeed only with $13=f$, in which case, $g=69$, and thence $\frac{1}{2}(f+g)=41$, and $\frac{1}{2}(f-g)=-28$, as before found.

An investigation for a different answer.

We have, on the first page,

$$a=(3rq-3ps)(r^2+3s^2),$$
$$b=(pr+3qs)(r^2+3s^2)-(p^2+3q^2)^2,$$
$$c=(3rq-3ps)(p^2+3q^2), \text{ and}$$
$$d=(r^2+3s^2)^2-(pr+3qs)(p^2+3q^2).$$

Let us now make r^2+3s^2 divisible by p^2+3q^2. This is done by putting $r=3n'q+m'p$, and $s=m'q-n'p$; for then r^2+3s^2 becomes $(m'^2+3n'^2)(p^2+3q^2)$. By substitution, we obtain

$$a=3n'(m'^2+3n'^2)(p^2+3q^2)^2,$$
$$b=m'(m'^2+3n'^2)(p^2+3q^2)^2-(p^2+3q^2)^2,$$
$$c=3n'(p^2+3q^2)^2, \text{ and}$$
$$d=(m'^2+3n'^2)^2(p^2+3q^2)^2-m'(p^2+3q^2)^2.$$

Dividing now by the common factor $(p^2+3q^2)^2$, we have

$$a=3n'(m'^2+3n'^2)=3gh(f^2+3g^2),$$
$$b=m'(m'^2+3n'^2)-1=fh(f^2+3g^2)-h^4,$$
$$c=3n'=3gh^3, \text{ and}$$
$$d=(m'^2+3n'^2)^2-m'=(f^2+3g^2)-fh^3;$$

that is, after putting $m'=f\div h$, $n'=g\div h$, and multiplying by h^4.

Whence, (and changing signs for positive numbers) we have the following theorems:

$$a+b=v=h(3g-f)(f^2+3g^2)+h^4,$$
$$a-b=z=h(3g+f)(f^2+3g^2)-h^4,$$

$$c+d=w=h^3(3g-f)+(f^2+3g^2)^2,$$
$$c-d=y=h^3(3g+f)-(f^2+3g^2)^2.$$

Beginning with the least numbers, we soon find that with $f=5$, $g=8$, and $h=14$, the least values of v, z, w, y, to answer the question, will result. Then v, z, w, y, will be, respectively, 96138, 49686, 99225, 32487; which, by dividing by the common factor $147=(0^2+3.7^2)$, become 654, 338, 675, 221; so that $(654)^3+(338)^3=(675)^3+(221)^3$. These are smaller numbers than those before found, within the relation to produce affirmative results. But, in attempting to find two *different* numbers, u and x, such that u^3+x^3 = the same sum, by either of the methods herein before employed successfully, we fail. We are, therefore, in this instance, driven to the following prolix, but equally well-known method:

It is known that $a^3+b^3=\left(\frac{a(a^3+2b^3)}{a^3-b^3}\right)^3-\left(\frac{b(2a^3+b^3)}{a^3-b^3}\right)^3$; in which, substituting $a=675$, and $b=221$, we obtain

$$(675)^3+(221)^3=\left(\frac{222165852975}{296753014}\right)^3-\left(\frac{138321162031}{296753014}\right)^3=a'^3-b'^3;$$

and from another equally well-known Theorem, we next have

$$u^3+x^3=\left(\frac{b'(2a'^3-b'^3)}{a'^3+b'^3}\right)^3+\left(\frac{a'(a'^3-2b'^3)}{a'^3+b'^3}\right)^3,$$ wherein

$$a'=\frac{222165852975}{296753014},\ b'=\frac{138321162031}{296753014}.$$ Wherefore,

$$u=\frac{b'(2a'^3-b'^3)}{a'^3+b'^3}=\frac{2667483866296141146514629941994412921894289729}{4039417275545528891152791084460320436090324},$$

$$x=\frac{a'^3(a'^3-2b^3)}{a'^3+b'^3}=\frac{1260271634077571065592737296883736641712234175.}{4039417275545528891152791084460320436090324}$$

We have now the numeral values of u, v, w, x, y, z, such that $v^3+z^3=w^3+y^3=u^3+x^3$; and, at the same time, those of x, y, z, are such as to insure all the numbers, in answer to the proposed question, to be affirmative. Multiplying all the numbers by the common denominator, in the above fractional values of u and x, we get integral values, as follows:

$u=2667483866296141146515629941994412921894289729,$
$v=2641778898206775894813925369237049565203071896,$
$w=27266066660993232001528133982010716294460968700,$
$x=1260271634077571065592737296883736641712234175,$
$y=892711217895561884944766829665730816375961604,$
$z=1365323039134388765209643386547588307398529512.$

These numbers are believed to be prime to each other, and are, therefore, irreducible by any common factor; and, if substituted in the notation with which we set out, namely:

$$\tfrac{1}{2}(x^3+y^3-z^3),\ \tfrac{1}{2}(x^3-y^3+z^3),\ \tfrac{1}{2}(-x^3+y^3+z^3),\ v^3-\tfrac{1}{2}(x^3+y^3-z^3),$$

or (because y is *odd*) in the equivalent expressions

$$4(x^3+y^3-z^3),\ 4(x^3-y^3+z^3),\ 4(-x^3+y^3+z^3),\ 8v^3-4(x^3+y^3-z^3),$$

will give four numbers such as are required in this problem.

2. Find three integral numbers in arithmetical progression, such that their common difference shall be a cube; the sum of any two, diminished by the third, a square; the sum of the roots of the required squares an 8th power; the first of the required squares a 7th power, the second a 5th power, the third a biquadrate, and the mean of the three required numbers a square.

Solution. Let $[(x^2-x+1)y^2,\ (x^2+1)y^2, \text{ and } (x^2+x+1)y^2] \ldots [a]$, represent the three numbers in arithmetical progression, their common difference being xy^2. Then must $\{ (x^2-2x+1)y^2=\square \ldots [1]$, $(x^2+1)y^2=\square \ldots [2]$, and $(x^2+2x+1)y^2=\square \ldots [3] \} \ldots [A]$. [1] and [3] are squares. To make [2] a square, let $x^2+1=(x-p)^2$; then will $x=(p^2-1)\div 2p$. This value of x in $[A]$ changes it to $\left\{ \left(\frac{p^2-2p-1}{2p}\right)^2 y^2 \ldots [4],\ \left(\frac{p^2+1}{2p}\right)^2 y^2 \ldots [5],\ \left(\frac{p^2+2p-1}{2p}\right)^2 y^2 \ldots [6] \right\} \ldots [B]$, all squares; the sum of their roots being $\left(\frac{3p^2-1}{2p}\right) y \ldots [7]$. Let $y=pz^2$, then $[B]$ and [7] become $\left\{ \left(\frac{p^2-2p-1}{2}\right)^2 z^4 \ldots [8],\ \left(\frac{p^2+1}{2}\right)^2 z^4 \ldots [9],\ \left(\frac{p^2+2p-1}{2}\right)^2 z^4 \ldots [10], \text{ and } \left(\frac{3p^2-1}{2}\right) z^2 \ldots [11] \right\} \ldots [C]$. [11] is a square and [10] a biquadrate when $p=1$. Putting $q+1$ for p, [10] and [11] will be $(\frac{q^2}{2}+2q+1)^2 z^4 \ldots [12]$, and $(\frac{3q^2}{2}+3q+1)^2 z^2 \ldots [13]$.

Assume $\frac{q^2}{2}+2q+1=(qr-1)^2$, then $q=\frac{4(r+1)}{2r^2-1}$. Substituting this value of q in [13], we obtain $\frac{24(r+1)^2}{(2r^2-1)^2}+\frac{12(r+1)}{2r^2-1}+1$.

Adding these terms, and rejecting the square denominator, the result is $4r^4+24r^3+44r^2+36r+13$, which make $=(2r^2+6r+2)^2$; then will $r=-\frac{3}{4}$, $q=\frac{4(r+1)}{2r^2-1}=8$, and $p=q+1=9$.

Substituting this value of p in $[C]$, we shall have $\{ 31^2z^4 \ldots [14],\ 41^2z^4 \ldots [15],\ 7^4z^4 \ldots [16],\ 11^2z^2 \ldots [17] \} \ldots [D]$. Take $z=11^3v^4$, and $[D]$ becomes $\{ 31^2.11^{12}v^{16} \ldots [18],\ 41^2.11^{12}v^{16} \ldots [19],\ 7^4.11^{12}v^{16} \ldots [20],\ 11^8v^8 \ldots [21] \} \ldots [E]$, and the common difference, $xy^2=360.11^{12}v^{16} \ldots [22]$. To make [22] a cube, let $v=5^2.3w^3$; then $[E]$ and [22] change to $\{ 31^2.11^{12}.5^{32}.3^{16}w^{48} \ldots [23],\ 41^2.11^{12}.5^{32}.3^{16}w^{48} \ldots [24],\ 7^4.11^{12}.5^{32}.3^{16}w^{48} \ldots [25],\ 11^8.5^{16}.3^8w^{24} \ldots [26],\ 11^{12}.5^{33}.3^{18}.2^3w^{48} \ldots [27] \} \ldots [F]$. [23] is a 7th power when $w=31^2.11^5.5^4.3^2u^7$, which changes $[F]$ to $\{ 31^{98}.11^{252}.5^{224}.3^{112}u^{336} \ldots [28],\ 41^2.31^{96}.11^{252}.5^{224}.3^{112}u^{336} \ldots [29],\ 7^4.31^{96}.11^{252}.5^{224}.3^{112}u^{336} \ldots [30],\ 31^{48}.11^{128}.5^{112}.3^{56}u^{168} \ldots [31],\ 31^{96}.11^{252}.5^{225}.3^{114}.2^3u^{336} \ldots [32] \} \ldots [G]$.

To make [29] a 5th power, put $u=41^3.31^4.11^3.5.3^3t^5$, and the expressions in $[G]$ become, respectively,

$$(41^{504}.31^{721}.11^{630}.5^{280}.3^{560}t^{840})^2=(41^{144}.31^{206}.11^{180}.5^{80}.3^{160}t^{240})^7,$$
$$(41^{505}.31^{720}.11^{630}.5^{280}.3^{560}t^{840})^2=(41^{202}.31^{288}.11^{252}.5^{112}.3^{224}t^{336})^5,$$
$$(41^{504}.31^{720}.11^{630}.7^2.5^{280}.3^{560}t^{840})^2=(41^{252}.31^{360}.11^{315}.7.5^{140}.3^{280}t^{420})^4,$$

$$41^{504}.31^{720}.11^{632}.5^{280}.3^{560}t^{840}=(41^{63}.31^{90}.11^{79}.5^{35}.3^{70}t^{105})^8,$$
$$41^{1008}.31^{1440}.11^{1260}.5^{561}.3^{1122}.2^3t^{1680}=(41^{336}.31^{480}.11^{420}.5^{187}.3^{374}.2t^{560})^3.$$

In the last five lines above, *eight* of the *nine* required powers are found; it remains to determine the required numbers; the *mean* of which will be the ninth of the required powers.

Since $x=(p^2-1)\div 2p$, $p=9$, and $y=pz^2$, the expressions in [**a**] are $1321z^4$, $1681z^4$, and $2041z^4$. But $z=11^3v^4$ changes them to $(1321.11^{12}.v^{16},\ 1681.11^{12}v^{16},\ 2041.11^{12}v^{16})$ . . . [H]; $v=5^2.3w^3$ makes these $1321.11^{12}.5^{32}.3^{16}w^{48}$, $1681.11^{12}.5^{32}.3^{16}w^{48}$, and $2041.11^{12}.5^{32}.3^{16}w^{48}$; and $w=31^2.11^5.3^2.5^4u^7$, transforms these last to

$$1321.31^{96}.11^{252}.5^{224}.3^{112}u^{336},$$
$$1681.31^{96}.11^{252}.5^{224}.3^{112}u^{336},$$
$$2041.31^{96}.11^{252}.5^{224}.3^{112}u^{336}.$$

Finally, $u=41^3.31^4.11^3.5.3^3t^5$; hence, the numbers sought are

$$1321.41^{1008}.31^{1440}.11^{1260}.5^{560}.3^{1120}t^{1680}=1321(41^7.31^{10})^{144}.(11^9.5^4.3^8t^{12})^{140},$$
$$(41^{505}.31^{720}.11^{630}.5^{280}.3^{560}t^{840})^2=1681(41^7.31^{10})^{144}.(11^9.5^4.3^8t^{12})^{140},$$
$$2041.41^{1008}.31^{1440}.11^{1260}.5^{560}.3^{1120}t^{1680}=2041(41^7.31^{10})^{144}.(11^9.5^4.3^8t^{12})^{140};$$

the numbers being simplest when $t=1$.

Note. [10] may be made a 4th power and [11] a square by other methods. If the quantities within the parentheses be multiplied by 4, the result is $2p^2+4p-2=\square$ . . . [a], and $6p^2-2=\square$ . . . [b]. Let $q+1=p$; then [a] and [b] become

$2q^2+8q+4=\square=b^2$. . . [a′], and $6q^2+12q+4=\square=a^2$ . . . [b′].

Subtracting [a′] from [b′], and factoring the difference, we obtain

$q(4q+4)=(a-b)(a+b)$. Take $q=a-b$, and $4q+4=a+b$.

Adding these two equations we have $a=(5q+4)\div 2$. Substituting this value of a in (b′), and reducing, we find $q=8$, and thence $p=9$.

To find a third value of p, put $s+9=p'$; then the quantities in the parentheses of [10] and [11] become

$2s^2+40s+169=\square=b'^2$ . . . [d], and $6s^2+108s+484=\square=a'^2$ . . . [c].

Multiplying [d] by 11^2, and [c] by 7^2, to make the numeral squares 169 and 484 equal, we obtain

$242s^2+4840s+23716=d^2$ . . . [d′], $294s^2+5292s+23716=c^2$ . . . [c′].

Subtracting [d′] from [c′], and factoring the difference, we have $s(52s+452)=(c-d)(c+d)$. Put $s=c-d$, and $52s+452=c+d$; then, by subtraction, $d=(51s+452)\div 2$. Substituting this value of d in [d′], and reducing, we find $p'=\frac{1017}{1633}$.

3. It is required to find three whole numbers in arithmetical progression, such that their common difference shall be a cube; the sum of any two, diminished by the third, a square; and the sum of the roots of these squares a square.

Solution. Let x^2-xy+y^2, x^2+y^2, and x^2+xy+y^2 represent the numbers in arithmetical progression, whose common difference is xy; then must

$x^2-2xy+y^2=\square$ . . . [1], $x^2+y^2=\square$ . . . [2], and $x^2+2xy+y^2=\square$ . . . [3].

[1] and [3] are already squares, and [2] is a square when $x=r^2-s^2$ and $y=2rs$, y being taken $>x$. Whence, by substitution, [1], [2], and [3], are changed to

$$(-r^2+2rs+{}^2s)^2,\ (r^2+s^2)^2, \text{ and } (r^2+2rs-s^2)^2;$$

the sum of whose roots, $r^2+4rs+s^2$ . . . [4], must be a square.

Put $r^2+4rs+s^2=(r+ns)^2$; then we find $r=\frac{(n^2-1)s}{4-2n}$.

Take $n=\frac{3}{2}$; then $r=\frac{5s}{4}$, $x=\frac{9s^2}{16}$, $y=\frac{10s^2}{4}$, and $xy=\frac{90s^4}{64}=\frac{3^2.10s^4}{2^6}=$ a cube . . . [5]; and the assumed numbers become, respectively,

$$\frac{1321s^4}{2^8} \ldots [6],\ \frac{1681s^4}{2^8} \ldots [7], \text{ and } \frac{2041s^4}{2^8} \ldots [8].$$

But [5] must be a cube, which is the case when $s=3.10^2t^3$, for then $xy=\frac{3^6.10^9t^{12}}{2^6}=\left(\frac{3^2.10^3t^4}{2^2}\right)^3=$ a cube; and [6], [7], and [8], become

$1321.5^8.3^4t^{12}=41797265625t^{12}$,
$1681.5^8.3^4t^{12}=53187890625t^{12}$,
$2041.5^8.3^4t^{12}=64578515625t^{12}$; where t may be any integer— the least numbers being when $t=1$.

If $s=2^3.3.10^2t^{12}$, or $t=2^{12}v^{12}$, the above numbers become

$1321.10^8.6^4v^{12}=171201600000000v^{12}$,
$1681.10^8.6^4v^{12}=217857600000000v^{12}$,
$2041.10^8.6^4v^{12}=264513600000000v^{12}$.

Substituting the values of r, x, and y, in terms of s, and taking $s=11^3.2^2v^4$, expressions [1], [2], [3], [4], [5], become [18] to [22], inclusive, and [6], [7], and [8], become the three expressions in [*H*], in the solution of problem 2.

Again, take $1321.5^8.3^4t^{12}$, $1681.5^8.3^4t^{12}$, and $2041.5^8.3^4t^{12}$, for the three squares sought, in the second condition of problem 2.

Then must $31^2.5^8.3^4t^{12}=$ a 7th power . . . (1),
$41^2.5^8.3^4t^{12}=$ a 5th power . . . (2),
$49^2.5^8.3^4t^{12}=$ a 4th power . . . (3), and the sum of the roots of these squares, $11^2.5^4.3^2t^{12}=$ an 8th power . . . (4).

(3) is already a 4th power, and expunging the 7th power factors from (1), the 5th power factors from (2), and extracting the square root of (4), we have only to make $31^2.5.3^4t^5=$ a 7th power . . . (5), $41^2.5^3.3^4t^2=$ a 5th power . . . (6), and $11.5^2.3t^3=$ a biquadrate . . . (7).

(7) is solved by taking $t=11.5^2.3u^4$. Whence, by substitution, (5) becomes $31^2.11^5.5^{11}.3^9u^{20}=$ a 7th power, and (6) becomes $41^2.11^2.5^7 3^6u^8=$ a 5th power, or, expunging the 7th and 5th power factors, as before, $31^2.11^5.5^4.3^2u^6=$ a 7th power . . . (8), and $41^2.11^2.5^2.3u^3=$ a 5th power . . . (9). Take $u=31^2.11^5.5^4.3^2v^7$, and (8) is satisfied. (9) then becomes $41^2.31^6.11^{17}.5^{14}.3^7v^{21}=$ a 5th power, or, cancelling 5th power factors $41^2.31.11^2.5^4.3^2v=$ a 5th power . . . (10). Hence $v=41^3.31^4.11^3.3^3w^5$ solves (10).

Retracing, we have $v^7=41^{21}.31^{28}.11^{21}.5^7.3^{21}w^{35}$,

$\therefore\ u=41^{21}.31^{30}.11^{26}.5^{11}.3^{23}w^{35}$, $u^4=41^{84}.31^{120}.11^{104}.5^{44}.3^{92}w^{140}$,
$\therefore\ t=41^{84}.31^{120}.11^{105}.5^{46}.3^{93}w^{140}$,
$t^{12}=41^{1008}.31^{1440}.11^{1260}.5^{552}.3^{1116}w^{1680}$,
$5^8.3^4t^{12}=41^{1008}.31^{1440}.11^{1260}.5^{560}.3^{1120}w^{1680}=m$,
$\therefore$ $1321m$, $1681m$, and $2041m$, satisfy all the conditions.

4. To find n consecutive numbers of the natural series, such that the sum of their cubes shall itself be a cube, n being $=$ a cube.

Solution. Let x represent the first number in the series, then must $x^3+(x+1)^3+(x+2)^3+(x+3)^3+$ &c. to $(x+n-1)^3=$ cube.

To obtain a more manageable expression for the sum of this series, we know that $1^3+2^3+3^3+$ &c. to $n^3=\frac{n^4+2n^3+n^2}{4}=\left(\frac{n(n+1)}{2}\right)^2$;

and, $\therefore$, that $1^3+2^3+3^3+$ &c. to $(x-1)^3=\left(\frac{(x-1)x}{2}\right)^2$; and

hence, that $1^3+2^3+3^3+$ &c. to $(x+n-1)^3=\left(\frac{(x+n-1)(x+n)}{2}\right)^2$.

Wherefore, the sum of all the cubes of the required series of n Nos. will be represented by $\left(\frac{(x+n-1)(x+n)}{2}\right)^2-\left(\frac{(x-1)x}{2}\right)^2=a^3$, cube.

$$\therefore\ (x+n-1)^2(x+n)^2-(x-1)^2x^2=4a^3.$$

But the difference of two squares is equal to the product of the sum and difference of the roots; therefore,

$$[(x+n-1)(x+n)+(x-1)x].[(x+n-1)(x+n)-(x-1)x]=4a^3.$$

$$\text{Or, } [2x^2+2(n-1)x+(n-1)n](2x+n-1)n=4a^3.$$

Here, n is, evidently, a factor of a^3; and since n is a cube number, $=t^3$, t is a factor of a; and we may represent $4a^3$ by $4b^3t^3$, and substituting t^3 for n, and dividing the equation by t^3, it becomes

$$[2x^2+2(t^3-1)x+(t^3-1)t^3](2x+t^3-1)=4b^3.$$

Making the multiplication in the left-hand member, and also multiplying both sides by 2, the equation is

$$8x^3+12(t^3-1)x^2+4(t^3-1)(2t^3-1)x+2(t^3-1)^2t^3=8b^3=\text{cube}.$$

Now this being one of those cubic formulas, irreducible by means of an assumption that will destroy *two terms*, we must avail ourselves of the next expedient, viz., the vanishing only of the term $8x^3$; and, in accomplishing this, in order that the utmost advantage of a division by factors may be had, that is, that the resulting equation may be divisible by $(t-1)(t+1)$, let us assume

$$2b=2x+[(t+1)^2+1](t-1);\ \text{then will}$$

$8x^3+12(t^2+2t+2)(t-1)x^2+(t^2+2t+2)^2.6(t-1)^2x+(t^2+2t+2)^3(t-1)^3=8b^3$. Equating these two equal cubic expressions, rejecting the common term $8x^3$, and transposing, we have

$$12[(t^2+2t+2)(t-1)-(t^3-1)]x^2+2[(t^2+2t+2)^2.3(t-1)^2-(t^3-1).2(2t^3-1)]x$$
$$=2(t^3-1)^2t^3-(t^2+2t+2)^3(t-1)^3.$$

Or, $12(t^2-1)x^2+2[(t^2+2t+2)^2.3(t-1)^2-(t^3-1).2(2t^3-1)]x=2(t^3-1)^2t^3-(t^2+2t+2)^3(t-1)^3$. Obviously, this equation is divisible by $t+1$, and also by $t-1$, and it reduces to

$$12x^2-2(t^4-6t^3-2t^2+10)x=t^7-3t^6-2t^5-2t^4+10t^3+4t^2-8.$$

Multiplying by 12, and adding $(t^4-6t^3-2t^2+10)^2$ to both sides, the quantity on the left-hand side will necessarily be a complete square; and the result will be

$$[12x-(t^4-6t^3-2t^2+10)]^2=t^8-4t^6+8t^2+4=(t^4-2t^2-2)^2,$$

$$\therefore\quad 12x-(t^4-6t^3-2t^2+10)=t^4-2t^2-2,\ \text{nad}$$

$x=\frac{1}{6}(t^4-3t^3-2t^2+4)$; where $t^3=n$, or n must be a cube number.

5. To find a cube number of numbers which are cubes whose roots are consecutive numbers of the natural series.

Solution. Let $(x-[n-1])^3$, $(x-[n-2])^3$, $(x-[n-3])^3 \dots x^3 \dots (x+[n-3])^3$, $(x+[n-2])^3$, $(x+[n-1])^3$, be an odd series of cube numbers whose roots are consecutive numbers of the natural series; x^3 being the middle term.

Then, beginning at this term, the sum of one term is x^3.

Also, the sum of three terms is $(x-1)^3+x^3+(x+1)^3=3x^3+6x$; sum of five terms is $(x-2)^3+(x-1)^3+x^3+(x+1)^3+(x+2)^3=5x^3+30x$

In like manner, the sum of seven terms is $=7x^3+84x$

and the sum of nine terms is $=9x^3+180x$

" " " " eleven terms is $=11x^3+330x$

" " " " thirteen " " $=13x^3+546x$, &c.

$\therefore$ the sum of $2n-1$ terms is $=(2n-1)x^3+(2n^3-3n^2+n)x \dots$ [1], for the nth term of the series 1, 3, 5, 7, 9, 11, 13, &c., is $2n-1$, and the $(n-1)$th term of the series 6, 30, 84, 180, 330, 546, &c., is $2n^3-3n^2+n$. But the problem requires that [1] shall be a cube, and also that $2n-1$ shall be a cube. Let $2n-1=p^3$; then $n=\frac{1}{2}(p^3+1)$, and by substitution, formula [1] becomes

$$p^3x^3+\frac{p^3(p^6-1)}{4}x=\text{a cube; or, dividing by } p^3,\ x^3+\frac{p^6-1}{4}x=\text{a cube.}$$

This expression will be a cube when $x=\frac{1}{2}$. Let $x=y+\frac{1}{2}$, then by substitution, multiplying by 8, and arranging the terms, we have $8y^3+12y^2+(2p^6+4)y+p^6=$a cube, which put$=(2y+p^2)^3$. Whence,

$$y=\frac{p^6-3p^4+2}{6(p^2-1)}=\frac{(p^2-1)^2-3}{6},\ \&\ x=\frac{(p^2-1)^2}{6}=\frac{(p-1)^2(p+1)^2}{6}.$$

Here the value of x must be integral, to have consecutive numbers. This will be effected by taking $p=6m\pm1$; m being any number.

Let $m=1$; then, using the negative sign, $p=5$, $x=96$, and $n=63$. Substituting these values in the original series, we have 34, 35, 36, ... 156, 157, 158, for the roots of 125 consecutive cubes of the natural series of nos. Using the plus sign, $p=7$, then $x=384$, $n=172$, and by substitution we have 213 .... 555, for the roots of 343 cubes which fulfill the conditions. If $m=2$, then using the msnus sign, $p=11$, we have 1331 cubes: and using the plus sign, $p=13$, and we shall have 2197 cubes. Now to find an even number of cubes, we have only to add to formula [1] the term in the original series next to $[x+(n-1)]^3$, viz., $(x+n)^3$, and then we shall have $(2n-1)x^3+(2n^3-3n^2+n)x+(x+n)^3=2nx^3+3nx^2+(2n^3+n)x+n^3=$ a cube. But $2n$ must also be a cube. Let $2n=p^3$; then $n=\frac{1}{2}p^3$, and by substitution, we have $p^3x^3+\frac{3p^3}{2}x^2+\frac{p^9+2p^3}{4}x+\frac{p^9}{8}=$a cube, which, after dividing by p^3, and multiplying by 8, becomes $8x^3+12x^2+(2p^6+4)x+p^6=$ a cube, which let$=(2x+p^2)^3$. Reducing this we find $x=\frac{p^6-3p^4+2}{6(p^2-1)}=\frac{(p^2-1)^2-3}{6}=\frac{(p-1)^2(p+1)^2-3}{6}$, where, in order to have x integral, p may be taken any even number, except 6, and its multiples. Let $p=2$; then $x=1$, $n=4$. Substituting these values in the original series, we have -2, -1, 0, 1, 2, 3, 4, 5, four of which numbers balance one another, and one is 0; whence we have 3, 4, 5, the roots of three cubes whose sum is 6^3.

Let $p=4$; then $x=37$, $n=32$, and by substitution we have 6, 7, 8, 9, ... 66, 67, 68, 69, for the roots of 64 cubes answering the conditions of the problem. Let $p=8$; then $x=661$, $n=256$, and we have the series 406, 407, ... 917, for the roots of 512 consecutive cubes of the natural series of numbers. Let $p=10$; then $x=1633$, $n=500$, and by substitution we have 1134, 1135, 1136, ... 2132, 2133, for the roots of 1000 cubes that fulfill the conditions.

6. To find n square numbers such, that if each be either increased or diminished by its root multiplied by some number, the respective sums and differences shall be squares.

Solution. We have here to make $x^2 \pm ax = \square$, $y^2 \pm by = \square$, $z^2 \pm cz = \square$, &c. ad infinitum. Let $x^2 + ax = \square = m^2x^2$; then will $x = \frac{a}{m^2 - 1}$, and, by substitution, $x^2 - ax = \frac{a^2}{(m^2-1)^2} - \frac{a^2}{m^2-1} = \square$. Dividing this by a^2, and multiplying by $(m^2-1)^2$, we have $2 - m^2 = \square$, which is so when $m = \pm 1$. Let $m = n - 1$, then $2 - m^2 = 1 + 2n - n^2 = \square = \left(1 - \frac{p}{q}n\right)^2$. Reducing, we find $n = \frac{2q(p+q)}{p^2+q^2}$; whence, $m = \frac{q^2 + 2pq - p^2}{p^2+q^2}$, and $x = \frac{a}{m^2-1} = \frac{(p^2+q^2)^2 a}{4pq(q^2-p^2)}$; whence, a may be taken $= 4pq(q^2 - p^2)$, and then $x = (p^2+q^2)^2$. Let $p = 1$ and $q = 2$; then $a = 24$, $x = 25$; whence, $x^2 \pm ax = 625 \pm 600 = 35^2$, or $5^2 = \square$'s.

Again: let $p = 2$, $q = 3$, then $a = 120$, $x = 169$, which let $= b$, and y, respectively; whence, $y^2 \pm by = 28561 \pm 20280 = 221^2$, or $91^2 = \square$'s.

Also, let $p = 1$, $q = 4$, then $a = 240$, $x = 289$, which let $= c$, and z, respectively; whence, $z^2 \pm cz = 83521 \pm 69360 = 391^2$, or $119^2 = \square$'s, and so on, ad infinitum.

7. To find a *common* value of x that will make $x^2 \pm ax = \square$, $x^2 \pm bx = \square$, $x^2 \pm cx = \square$, &c., ad infinitum.

Solution. In order to solve this problem, it will be necessary to premise the following Algebraic Theorems from Barlow's Theory of Numbers:

1. Every prime number of the form $4t + 1$, is the sum of two squares in one way only.

2. The square of every such number is also the sum of two $\square$'s in one way only.

3. The product of any two such numbers is the sum of two squares in two ways; the product of any three such numbers is the sum of two squares in four ways; the product of any four such numbers is the sum of two squares in eight ways; and, generally, the product of any (n) such numbers is the sum of two squares in 2^{n-1} ways. By attending to these theorems, the following solution will be readily understood.

Put $x = z^2$; then by substitution, and expunging z^2, we have

$$z^2 \pm a = \square,\ z^2 \pm b = \square,\ z^2 \pm c = \square,\ \text{\&c. ad infinitum.}$$

Let $z^2 = m^2 + n^2$, and $a = 2mn$; then $z^2 \pm a = m^2 \pm 2mn + n^2 = (m \pm n)^2$. Also let $z^2 = p^2 + q^2$, and $b = 2pq$; then $z^2 \pm b = p^2 \pm 2pq + q^2 = (p \pm q)^2$. Again, let $z^2 = r^2 + s^2$, and $c = 2rs$; then $z^2 \pm c = r^2 \pm 2rs + s^2 = (r \pm s)^2$; all $\square$'s; and so on, ad infinitum. Here all the conditions of the problem will be satisfied, if we can divide z^2 into two squares in any n number of ways. In order to do this, it will be necessary to solve the following problem: "To divide a given square number into two other square numbers". See Barlow, p. 460.

Let $A^2 = z^2 =$ the given square, and v^2, w^2, be the required squares; then $A^2 = v^2 + w^2$, and $A^2 - w^2 = v^2$. Assume $A + w = \frac{p'v}{q'}$, and $A -$

$w=\frac{q'v}{p'}$. Hence we readily find $2A=\frac{p'v}{q'}+\frac{q'v}{p'}=\frac{(p'^2+q'^2)v}{p'q'}$, and $2w=\frac{p'v}{q'}-\frac{q'v}{p'}=\frac{(p'^2-q'^2)v}{p'q'}$; and thence we get $v=\frac{2p'q'A}{p'^2+q'^2}$, $w=\frac{(p'^2-q'^2)A}{p'^2+q'^2}$; in which formulæ, A, or its equal, z, may be taken any prime number, or product of two or more prime numbers, each of which is the sum of two squares; then $p'^2+q'^2$ being put successively equal to each sum of two squares into which A or z can be divided, we shall have as many integral values of v, w; whence also, $z^2=v^2+w^2$ the same number of different ways, for which may be put successively, m^2+n^2, p^2+q^2, r^2+s^2, &c. The different values of $2vw$ may be put successively$=2mn$, $2pq$, $2rs$, etc.,$=a$, b, c, etc.

Let $A=z=5=2^2+1^2$, and put $p'=2$, $q'=1$; then $v=4$, $w=3$, and $z^2=5^2=4^2+3^2$, being the sum of two squares in *one* way only.

Let $A=z=5.13=65$; then $5=2^2+1^2$, $13=3^2+1^2$, $65=8^2+1^2=7^2+4^2$, and putting p', q', successively=the root of each set of two squares, we shall have $v=52$, $w=39$, $v=60$, $w=25$, $v=16$, $w=63$, $v=56$, $w=33$; whence z^2=the sum of two squares in *four* ways.

Let $z=5.13.17=1105$; then $5=2^2+1^2$, $13=3^2+2^2$, $17=4^2+1^2$, $5.13=65=8^2+1^2=7^2+4^2$, $5.17=85=9^2+2^2=7^2+6^2$, $13.17=221=11^2+10^2=14^2+5^2$, $1105=33^2+4^2=32^2+9^2=31^2+12^2=24^2+23^2$; and putting p', q', successively=the roots of each set of two □'s, we shall have the following values of v, w, viz., 884, 663; 1020, 425; 520, 975; 272, 1071; 952, 561; 468, 1001; 1092, 169; 1100, 105; 700, 855; 264, 1073; 576, 943; 744, 817; 1104, 47: whence z^2=the sum of two squares in *thirteen* ways.

In like manner, if z=the product of four factors, each of which is the sum of two squares; then, taking the factors singly, the product of every two, the product of every three, and finally, the product of all four, we shall find *forty* ways in which z^2 will be divided into the sum of two squares; and if z=the product of five such factors, we shall, by proceeding as above, find *one hundred and twenty one* ways in which z^2 will be divided into the sum of two □'s.

Premising that the subscript figures represent the No. of factors used, we have the series 1_1, 4_2, 13_3, 40_4, 121_5, 364_6, 1093_7, 3280_8, &c., which may be extended ad infinitum, by multiplying any term in the series by 3, and adding 1, to find the next succeeding term.

Having thus found the values of z, a, b, c, &c., in the expressions $z^2\pm a=$□, $z^2\pm b=$□, $z^2\pm c=$□, &c, if we multiply these expressions by $x=z^2$, we shall solve the original expressions

$x^2\pm ax=$□, $x^2\pm bx=$□' $x^2\pm cx=$□, &c.

Note. If $z=5.5.13=325$, then $5=2^2+1^2$, $13=3^2+2^2$, $5.5=25=4^2+3^2$, $5.13=65=8^2+1^2=7^2+4^2$, and $5.5.13=325=18^2+1^2=17^2+6^2=15^2+10^2$. The last set is in the same ratio as 3^2+2^2, and will give the same values of v, w; therefore, there are but 7 ways in which z^2 can be divided into the sum of two squares.

We see from the above solution how to construct expressions of this kind to any extent we please, so as, having given the values of a, b, c, etc., to find z, and thence x.

8. It is required to find any n square numbers such, that if the root of each be either added to or subtracted from the respective squares, the sums and differences shall be squares.

Solution. Let $a^2x^2 \pm ax = \square$, $b^2y^2 \pm by = \square$, $c^2z^2 \pm cz = \square$, etc., ad infinitum. Here we will solve only the case $a^2x^2 \pm ax = \square$. Put $a^2x^2 + ax = \square = m^2x^2$, then reducing we find $x = \frac{a}{m^2 - a^2}$, which being put in the expression $a^2x^2 - ax = \square$, gives $\frac{a^4}{(m^2-a^2)^2} - \frac{a^2}{m^2-a^2} = \square$. Dividing this by a^2, and $\times$ing by $(m^2-a^2)^2$, we have $2a^2 - m^2 = \square$; which is so when $m = \pm a$. Let $m = n - a$, then, by substitution, we have $a^2 + 2an - n^2 = \square = \left(a^2 - \frac{p}{q}n\right)^2$; whence $n = \frac{a(2pq+2q^2)}{p^2+q^2}$, where a may be taken $= p^2+q^2$, and $n = 2pq + 2q^2$; whence $m = n - a$, or $a - n = p^2 - 2pq - q^2$, and $x =$

$$\frac{p^2+q^2}{4pq^3 - 4p^3q} = \frac{p^2+q^2}{4pq(q^2-p^2)}, \text{ and } ax = \frac{(p^2+q^2)^2}{4pq(q^2-p^2)}.$$

Now let $p=1$, $q=2$, then $a=5$; $x = \frac{5}{24}$, $ax = \frac{25}{24}$, $(\frac{25}{24})^2 \pm \frac{25}{24} = (\frac{35}{24})^2$ and $(\frac{5}{24})^2$. Again, let $p=2$, $q=3$; then $a=13$, $x = \frac{13}{120}$, which let $= b$ and y, respectively; then $by = \frac{169}{120}$, and $(\frac{169}{120})^2 \pm \frac{169}{120} = (\frac{221}{120})^2$, $(\frac{91}{120})^2$. Also, let $p=1$, $q=4$, then $a=17$; $x = \frac{17}{240}$, which let $= c$ and z, respectively; then $cz = \frac{289}{240}$, and $(\frac{289}{240})^2 \pm \frac{289}{240} = (\frac{391}{240})^2$, and $(\frac{119}{240})^2$, and so on.

9. To find a *common* value of x that will make $a^2x^2 \pm ax = \square$, $b^2x^2 \pm bx = \square$, $c^2x^2 \pm cx = \square$, etc., ad infinitum.

Solution. Dividing these expressions by a^2, b^2, c^2, etc., respectively, and writing a', b', c', etc., for the reciprocals of a, b, c, etc., respectively, we have $x^2 \pm a'x = \square$, $x^2 \pm b'x = \square$, $x^2 \pm c'x = \square$, etc.

These expressions are of the same form as those in problem 7, the only difference being that the values of a, b, c, etc., are integral in 7; and a', b', c', etc., in this, are fractional; x being integral in both.

Let $x = z^2$; then substituting and expunging z^2, we have

$$z^2 \pm a' = \square,\quad z^2 \pm b' = \square,\quad z^2 \pm c' = \square, \text{ etc., ad infinitum.}$$

Let $z^2 = m^2 + n^2$, and $a' = 2mn$; then $z^2 \pm a' = m^2 \pm 2mn + n^2 = (m \pm n)^2$.
Also let $z^2 = p^2 + q^2$, and $b' = 2pq$; then $z^2 \pm b' = p^2 \pm 2pq + q^2 = (p \pm q)^2$.
And let $z^2 = r^2 + s^2$, and $c' = 2rs$; then $z^2 \pm c' = r^2 \pm 2rs + s^2 = (r \pm s)^2$, etc.

Here, as in problem 7, all the conditions will be satisfied, if we can divide z^2 into two squares any (n) number of ways; a method of doing which has been given in the solution of problem 7.

Let us subjoin a few examples of problem 7; as this is easily changed to 9, they will illustrate both. See problems 11 and 12.

(**a**). Let $z = 5.5.13 = 325$; then, by Theorems on p. 10, we have $5 = 2^2 + 1^2$, $13 = 3^2 + 2^2$, $5.5 = 25 = 4^2 + 3^2$, $5.13 = 65 = 8^2 + 1^2 = 7^2 + 4^2$, $5.5.13 = 325 = 18^2 + 1^2 = 17^2 + 6^2 = 15^2 + 10^2$. This last set being in the same ratio as $3^2 + 2^2$, as before stated, gives the same values to v, w.

Let $p' = 2$, $q' = 1$; then $v = 260$, $w = 195$, and $2vw = 101400 = a$.
" $p' = 3$, $q' = 2$; " $v = 300$, $w = 125$, " $2vw = 75000 = b$.
" $p' = 4$, $q' = 3$; " $v = 312$, $w = 91$, " $2vw = 56784 = c$.
" $p' = 8$, $q' = 1$; " $v = 80$, $w = 315$, " $2vw = 50400 = d$.

" $p'=7$, $q'=4$; " $v=280$, $w=165$, " $2vw=92400=e$,
" $p'=18$, $q'=1$; " $v=36$, $w=323$, " $2vw=23256=f$,
" $p'=17$, $q'=6$; " $v=204$, $w=253$, " $2vw=103224=g$.

$\therefore$ $z^2 \pm a=(325)^2 \pm 101400=207025$ and 4225, or $(455)^2$, $(65)^2$,
$z^2 \pm b=(325)^2 \pm 75000=180625$ " 30625, " $(425)^2$, $(175)^2$,
$z^2 \pm c=(325)^2 \pm 56784=162409$ " 48841, " $(403)^2$, $(221)^2$,
$z^2 \pm d=(325)^2 \pm 50400=156025$ " 55225, " $(395)^2$, $(235)^2$,
$z^2 \pm e=(325)^2 \pm 92400=198025$ " 13225, " $(445)^2$, $(115)^2$,
$z^2 \pm f=(325)^2 \pm 23256=128881$ " 82369, " $(359)^2$, $(287)^2$,
$z^2 \pm g=(325)^2 \pm 103224=208849$ " 2401, " $(457)^2$, $(49)^2$.

By considering this solution we readily see how to construct the following problem:

(***b***). Make $x^2 \pm 101400x=\square$, $x^2 \pm 75000x=\square$, $x^2 \pm 56784x=\square$, $x^2 \pm 50400x=\square$, $x^2 \pm 92400x=\square$, $x^2 \pm 23256x=\square$, $x^2 \pm 103224x=\square$. We see also how to find the value of x, when a, b, c, &c., are given.

Take any of the given quantities, as, e. g., 23256; then we have $2vw=23256$, and $vw=11628=2.2.3.3.17.19$. Put $v=2.2.3.3=36$, & $w=17.19=323$; then $z^2=v^2+w^2=36^2+323^2=325^2$, which No. will satisfy all the conditions; whence $x=z^2$ is known.

(***c***). Let $z=5.5.17=425$; then $5=2^2+1^2$, $17=4^2+1^2$, $5.5=25=4^2+3^2$, $5.17=85=9^2+2^2=7^2+6^2$, $5.5.17=425=19^2+8^2=16^2+13^2=20^2+5^2$. The last set being in the same ratio as 4^2+1^2, will, as stated in the *Note* on page 11, give the same values of v, w.

Let $p'=2$, $q'=1$; then $v=340$, $w=255$ and $2vw=173400=a$,
$p'=4$, $q'=1$ " $v=200$, $w=375$ " $2vw=150000=b$,
$p'=4$, $q'=3$ " $v=408$, $w=119$ " $2vw=97104=c$,
$p'=9$, $q'=2$ " $v=180$, $w=385$ " $2vw=138600=d$,
$p'=7$, $q'=6$ " $v=420$, $w=65$ " $2vw=54600=e$,
$p'=19$, $q'=8$ " $v=304$, $w=297$ " $2vw=180576=f$,
$p'=16$, $q'=13$ " $v=416$, $w=87$ " $2vw=72384=g$.

$\therefore$ $z^2 \pm a=(425)^2 \pm 173400=354025$ and 7225, or $(595)^2$, $(85)^2$,
$z^2 \pm b=(425)^2 \pm 150000=330625$ " 30625, " $(575)^2$, $(175)^2$,
$z^2 \pm c=(425)^2 \pm 97104=277729$ " 83521, " $(527)^2$, $(289)^2$,
$z^2 \pm d=(425)^2 \pm 138600=319225$ " 42025, " $(565)^2$, $(205)^2$,
$z^2 \pm e=(425)^2 \pm 54600=235225$ " 126025, " $(485)^2$, $(355)^2$,
$z^2 \pm f=(425)^2 \pm 180576=361201$ " 49, " $(601)^2$, 7^2,
$z^2 \pm g=(425)^2 \pm 72384=253009$ " 108241, " $(503)^2$, $(329)^2$.

From this solution we easily construct the following problem:

(***d***). Make $x^2 \pm 173400x=\square$, $x^2 \pm 150000x=\square$, $x^2 \pm 97104x=\square$, $x^2 \pm 138600x=\square$, $x^2 \pm 54600=\square$, $x^2 \pm 180576x=\square$, and $x^2 \pm 72384x=\square$. We see, also, how to find the value of x, when a, b, c, etc., are given.

Take any of the given quantities, as, e. g., 97104; then $2vw=97104$, $vw=48552=2.2.2.3.7.17.17$. Put $v=2.2.2.3.17=408$, $w=7.17=119$; then $z^2=v^2+w^2=(408)^2+(119)^2=(425)^2$, which number satisfies all the conditions simultaneously; whence $x=z^2$ is known.

Hence, also, we see how to solve the problem,

(***e***). Make $x^2-(101400)^2=\square$, $x^2-(75000)^2=\square$, $x^2-(56784)^2=\square$.

Here, $x^2-(101400)^2=(x+101400)(x-101400)$, $x^2-(75000)^2=(x+75000)(x-75000)$, and $x^2-(56784)^2=(x+56784)(x-56784)$.

Let $x=z^2$; then it is evident that we have to make $z^2 \pm 101400=\square$, $z^2 \pm 75000=\square$, $z^2 \pm 56784=\square$; for if each factor is a $\square$, the product is also a $\square$. The problem is now the same as (***b***).

10. Find three square numbers in arithmetical progression, such that if from each its root be *subtracted*, the three remainders shall be rational squares.

HISTORY. **So far as we know, this problem was originally published in the first London edition of J. R. Young's Algebra; where erroneous answers were given. The American Editor, noticing this fact, omitted the problem; and in a note he says, "It would be difficult to find true numbers of any moderation, unless negative answers be admitted".**

Solution. Let a^2x^2, b^2x^2, c^2x^2, represent the numbers required. Then, by the conditions, $[a^2x^2-ax=\square$ .. (1), $b^2x^2-bx=\square$ . . . (2), and $c^2x^2-cx=\square$ . . . (3)] . . . (C). Put $a^2x^2-ax=m^2x^2$; then will

$x=\dfrac{a}{a^2-m^2}$ . . . (4). Putting this value of x in (2) and (3), and multiplying by $(a^2-m^2)^2$, they become, respectively,

$a^2b^2-(a^2-m^2)ab=\square$ . . . (5), and $a^2c^2-(a^2-m^2)ac=\square$ . . . (6). These are $\square$'s if $m=\pm a$, but either of these values of m renders x infinite. If $n\pm a$ be substituted for m in (5) and (6), we have $a^2b^2\pm2a^2bn+abn^2=\square$ . . . (7), and $a^2c^2\pm2a^2cn+acn^2=\square$ . . . (8). Take $ab-np$ for the root of (7), and we have $n=\dfrac{2ab(p\pm a)}{p^2-ab}$ . . . (9). Substituting this value of n in (8), and multiplying by $\left(\dfrac{p^2-ab}{a}\right)^2$ we obtain $c^2p^4\pm4abcp^3+2abc(2a+2b-c)p^2\pm4a^2b^2cp+a^2b^2c^2=\square$ . . . (10). Assuming $cp^2+2abp-abc$ for the root of (10), we have

$$p=\frac{2abc}{ab-ac-bc},\ \text{or}\ \frac{ac+bc-ab}{2c}\ \ldots\ (11).$$

But if $cp^2-2abp-abc$ be taken for the root of (10), we have

$$p=\frac{ab-ac-bc}{2c},\ \text{or}\ \frac{2abc}{ac+bc-ab}\ \ldots\ (11').$$

From (4), $x=\dfrac{a}{a^2-m^2}$, or, since $m^2=a^2\pm2an+n^2$, $x=-\dfrac{a}{n(n\pm2a)}$ . . . (12). If we substitute for n in (12), its value as shown in (9), we shall get

$$x=\frac{-(p^2-ab)^2}{4abp(p\pm a)(p\pm b)}\ \ldots\ (13).$$

If, in (13), we substitute for p its first value in (11), or that in (11′), using the $+$ sign in the binomial factors of the denominator, we shall find

$$x=\frac{-\{(ab-ac-bc)^2-4abc^2\}^2}{8abc(ac-ab-bc)(ab-ac+bc)(ab+ac-bc)}\ \ldots\ (14).$$

When the relation between $a, b, c,$ is such as to make the numerical value of x in (14) negative, (C) becomes $[a^2x^2+ax=\square$ . . . (1′), $b^2x^2+bx=\square$ . . . (2′), $c^2x^2+cx=\square$ . . . (3′)] . . . (C′); then

$$x=\frac{\{(ab-ac-bc)^2-4abc^2\}^2}{8abc(ab-ac-bc)(ab-ac+bc)(ab+ac-bc)}\ \ldots\ (15).$$

This shows that for such positive values of a, b, c, the conditions of (C) are impossible, by the general method, when positive anwers are required, and that (C) must be changed to (C′), that the problem may be possible; a^2x^2, b^2x^2, c^2x^2, being three square numbers, such, that if each be *added* to its respective root, the sums will be rational squares.

Whence, premising that A represents the square root of the numerator of (14), and B the product of the trinomial factors of the denominator, the required squares, in both cases, will be

$$a^2x^2 \pm ax = \left(\frac{A}{8bcB}\right)^2 (a^2b^2 - 2a^2bc - 2ab^2c - a^2c^2 + 2abc^2 - b^2c^2)^2,$$

$$b^2x^2 \pm bx = \left(\frac{A}{8acB}\right)^2 (a^2b^2 - 2a^2bc - 2ab^2c - 3a^2c^2 + 2abc^2 + b^2c^2)^2,$$

$$c^2x^2 \pm cx = \left(\frac{A}{8abB}\right)^2 (3a^2b^2 - 2a^2bc - 2ab^2c - a^2c^2 + 2ab^2c - b^2c^2)^2.$$

Again: if, in expression (13) we put for p either its second value in (11), or that in (11′), employing the — sign in the factors of the denominator, we obtain the same value for x; a, b, c, being any numbers whatever. Since a, b, c, may be interchanged without altering the value of x, it is evident that this process will give but one value of x that will fulfill the conditions in (C); but other values of p may be found by substituting

$q + \frac{ac+bc-ab}{2c}$, or $q + \frac{2abc}{ac+bc-ab}$, for p in (10), and making the result a square; the values of p, thus found, will produce other values of x that will fulfill the conditions in (C).

Since the numerator of x is a square, and the denominator divisible by a, b, c, the square of the numerator of x will be the common numerator of ax, bx, cx; therefore, when a^2x^2-ax, b^2x^2-bx, c^2x^2-cx, are squares, if the denominator of ax, bx, cx, be subtracted from the numerator, the remainder will be a square; but if a^2x^2+ax, b^2x^2+bx, c^2x^2+cx, are squares, the *sum* of the numerator and denominator of ax, bx, cx, will each be a square.

To fulfill the conditions of the problem, a^2x^2, b^2x^2, c^2x^2, or, expunging x^2, a^2, b^2, c^2, must be in arithmetical progression, and have such relative values as to make x positive.

$a^2=(p^2-2p-1)^2q^2$, $b^2=(p^2+1)^2q^2$, $c^2=(p^2-2p+1)^2q^2$, are in arithmetical progression; and to obtain the least numbers for the numerator and denominator in the fractional value of x, let $p=7$, and $q=\frac{1}{2}$; then $a=31$, $b=25$, and $c=17$. Or we may take $a^2=(r^2-2rs-s^2)^2$, $b^2=(r^2+s^2)^2$, $c^2=(r^2+2rs-s^2)^2$; in which the values of $r+s$, $r-s$, being substituted for r, s, give the same values of a, b, c, or the same multiple of those values; and any multiple of the values of a, b, c, will give the same value of x. If $r=4$, $s=3$, then will $a=-17$, $b=25$, $c=31$; but if $r=7$, and $s=1$, then will $a=34$, $b=50$, $c=62$. Substituting either of these three sets of values for a, b, c, in (14), we find $x=\frac{(864571)^2}{11011044931800}$;

$$\therefore\ a^2x^2=\frac{(864571)^4}{(647708525400)^2}, \text{ and } a^2x^2-ax=\left(\frac{864571\cdot315871}{647708525400}\right)^2;$$

$$b^2x^2=\frac{(864571)^4}{(440441797272)^2}, \quad b^2x^2-bx=\left(\frac{864571.554113}{440441797272}\right)^2;$$

$$c^2x^2=\frac{(864571)^4}{(355194997800)^2}, \quad c^2x^2-cx=\left(\frac{864571.626329}{355194997800}\right)^2.$$

These are the least numbers that have yet been found, when the roots, ax, bx, cx, are taken with the positive sign; when reduced to a common denominator they become

$$a^2x^2=\left(\frac{12707211238697}{11011044931800}\right)^2, \quad \text{and } a^2x^2-ax=\left(\frac{4642579407797}{11011044931800}\right)^2;$$

$$b^2x^2=\left(\frac{18687075351025}{11011044931800}\right)^2, \quad b^2x^2-bx=\left(\frac{11976750763075}{11011044931800}\right)^2;$$

$$c^2x^2=\left(\frac{23171973435271}{11011044931800}\right)^2, \quad c^2x^2-cx=\left(\frac{16786682585629}{11011044938100}\right)^2.$$

The product of the sum and difference of two quantities being equivalent to the difference of their squares, the numerator of (14) may be exhibited under a different form, which has the advantage of symmetry, of corresponding more nearly with the denominator, and of showing, at a glance, that a, b, c, may be interchanged, but the change of form introduces radicals; all of which, however, disappear upon performing the multiplications indicated. We have

$$(ab-ac-bc)^2-4abc^2=(ab-ac-bc+2c\sqrt{ab})(ab-ac-bc-2c\sqrt{ab})=$$
$$\{ ab-(ac-2c\sqrt{ab}+bc) \}.\{ ab-(ac+2c\sqrt{ab}+bc) \}=$$
$$\{ ab-(\sqrt{ac}-\sqrt{bc})^2 \}.\{ ab-(\sqrt{ac}+\sqrt{bc})^2 \}=(\sqrt{ab}-\sqrt{ac}+\sqrt{bc})$$
$$(\sqrt{ab}+\sqrt{ac}-\sqrt{bc})(\sqrt{ab}-\sqrt{ac}-\sqrt{bc})(\sqrt{ab}+\sqrt{ac}+\sqrt{bc}).$$

11. If, instead of three $\square$ Nos. in arith. prog., we should wish To find three $\square$ numbers, a^2x^2, b^2x^2, c^2x^2, such, that ($a^2x^2\pm ax=\square$, $b^2x^2\pm bx=\square$, $c^2x^2-cx=\square$) . . . [***b***], the value of x, in [14], fulfills three of these conditions; and [13] gives $ax=\frac{(p^2-ab)^2}{-4bp(a+p)(b+p)}$, the numerator of which being a square, we have only to make the sum of the numer. and denom. a $\square$, in order to have $a^2x^2-ax=\square$. Since the denominator is *negative*, the sum of these is found to be

$$p^4-4bp^3-6abp^2-4b^2p^2-4ab^2p+a^2b^2=\square \ . \ . \ . \ [16].$$

Taking $p^2-2bp-ab$ for the root of [16], $p=-\frac{2ab}{a+2b}$ . . . [17].

But from [11′], $p=\frac{ab-ac-bc}{2c}$, $\therefore -\frac{2ab}{a+2b}=\frac{ab-ac-bc}{2c}$,

and $c=\frac{ab(a+2b)}{a^2-ab+2b^2}$ . . . [18]. Putting this value of c for c in [14], multiplying by b, and dividing both numerator and denominator of the right-hand side of the resulting equation by $16a^6b^6$, we obtain

$bx=\frac{(a^2+4b^2)^2}{(a+2b)\times-4(-2a+4b^2)a^2}$. Since the numerator of this fraction is a square, we have only to make the difference between the numerator and denominator a $\square$, in order to make $b^2x^2+bx=\square$. This difference is $-7a^4+40a^2b^2+16b^4$, which must be a $\square$.

If $b=1$, this becomes $-7a^4+40a^2+16=\square$ . . . [19]. This is a $\square$ when $a=\pm2$; let, $\therefore$ $a=f+2$, then [19] will become $-7f^4-56f^3-128f^2-64f+64=\square$, which put $=(9f^2-4f+8)^2$; then $f=-\frac{16}{11}$, and $a=f+2=\frac{6}{11}$; but $b=1$, $\therefore$ $a : b :: \frac{6}{11} : 1 :: 6 : 11$. We may, therefore, take $a=6$, and $b=11$; then, from [18], $c=\frac{7.66}{53}$. Multiplying a, b, c, by 53, to obtain integral values, $a=6.53$, $b=11.53$, $c=7.66$. These values of a, b, c, in [14], give $x=\frac{8580^2}{20478915072}=\frac{715^2}{142214688}$.

But if $a=f-2$, [19] becomes $-7f^4+56f^3-128f^2+64f+64=\square$, which put$=(9f^2-4f-8)^2$, then $f=\frac{16}{11}$, $a=f-2=-\frac{6}{11}$. Therefore, $a : b :: -\frac{6}{11} : 1 :: -6 : 11$. If $a=-6$, $b=11$, and from [18] $c=-\frac{11 \cdot 12}{43}$. Multiplying a, b, c, by 43, we may take $a=-6.43$, $b=11.43$, $c=-11.12$, and from [17], $p=-\frac{3 \cdot 11 \cdot 43}{4}$. These values of a, b, p, in [13], give $x=\frac{65^2}{953568}$. Either of these two values of x fulfills the five conditions in [***b***].

Other values of x may be found by writing $v\pm\frac{6}{11}$ for a, in [19], and making the results squares. Similarly, others may be found.

12. If it were required to find a *common* value of x, so that $a^2x^2\pm ax=\square$, $b^2x^2\pm bx=\square$, $c^2x^2\pm cx=\square$, &c., ad infinitum, we remark first, that all the prime numbers contained in the formula $4t+1$, are each the sum of two squares; which call r^2+s^2; $r'^2+s'^2$; $r''^2+s''^2$, &c., successively, where r, s; r', s'; r'', s'', &c., may be any Nos. the sum of whose $\square$'s is a prime, derived from $4t+1$.

Then take $x'=[(r^2+s^2)(r'^2+s'^2)(r''^2+s''^2).\&c.].(r^2+s^2)^2$,

$$x''=[(r^2+s^2)(r'^2+s'^2)(r''^2+s''^2).\&c.].(r^2+s^2)^4,$$
$$x'''=[(r^2+s^2)(r'^2+s'^2)(r''^2+s''^2).\&c.].(r^2+s^2)^6, \text{ \&c., to}$$
$$[(r^2+s^2)(r'^2+s'^2)(r''^2+s''^2).\&c.].(r^2+s^2)^{2n}.$$

Taking the factors r^2+s^2, $r'^2+s'^2$, and putting $r=2$, $s=1$; $r'=3$, $s'=2$, these general expressions will be $x=(2^2+1^2)^2(3^2+2^2)^2$, $x'=65^2(2^2+1^2)^2$, $x''=65^2(2^2+1^2)^4$, $x'''=65^2(2^2+1^2)^6$, &c., to $65^2(2^2+1^2)^{2n}$.

As the no. of ways in which r^2+s^2 can be divided into the sum of two $\square$'s is *infinite*, we can find as many expressions like the first two in [***b***], p. 16, as we choose, all having a common value of x.

Since $65^2=(8^2+1^2)^2=(7^2+4^2)^2=5^2(3^2+2^2)^2=13^2(2^2+1^2)^2$, we have, for four sets of three square numbers in arithmetical progression,

$(8^2-2.8.1-1^2)^2$, $(8^2+1^2)^2$, $(8^2+2.8.1-1^2)^2$, com. diff.$=4.8.1(8^2-1^2)=a$,
$(7^2-2.7.4-4^2)^2$, $(7^2+4^2)^2$, $(7^2+2.7.4-4^2)^2$, " " $=4.7.4(7^2-4^2)=b$,
$5^2(3^2-2.3.2-2^2)^2$, $5^2(3^2+2^2)^2$, $5^2(3^2+2.3.2-2^2)^2$, " " $=5^2.4.3.2(3^2-2^2)=c$,
$13^2(2^2-2.2.1-1^2)^2$, $13^2(2^2+1^2)^2$, $13^2(2^2+2.2.1-1^2)^2$, " $=13^2.4.2.1(2^2-1^2)=d$.

Writing for a, b, c, d, the square of their reciprocals, we obtain

$$a^2x^2\pm ax=\square=\left(\frac{65^2}{2016}\right)^2\pm\frac{65^2}{2016}=\left(\frac{65.89}{2016}\right)^2, \text{ or } \left(\frac{65.23}{2016}\right)^2,$$
$$b^2x^2\pm bx=\square=\left(\frac{65^2}{3696}\right)^2\pm\frac{65^2}{3696}=\left(\frac{65.79}{3696}\right)^2, \text{ or } \left(\frac{65.47}{3696}\right)^2,$$
$$c^2x^2\pm cx=\square=\left(\frac{65^2}{3000}\right)^2\pm\frac{65^2}{3000}=\left(\frac{65.85}{3000}\right)^2, \text{ or } \left(\frac{65.35}{3000}\right)^2,$$
$$d^2x^2\pm dx=\square=\left(\frac{65^2}{4056}\right)^2\pm\frac{65^2}{4056}=\left(\frac{65.91}{4056}\right)^2, \text{ or } \left(\frac{65.13}{4056}\right)^2.$$

The above values of a, b, c, substituted in formula [14], page 14, will not give $x=65^2$; the value of x, thus obtained, will satisfy only three of the conditions in [***b***].

Because $m^2+n^2+2mn=\square$, and $m^2+n^2-2mn=\square$, we know that $m^2+n^2\pm2mn=\square$, which seems to be the base of the general problem to which these examples belong. Substituting r^2-s^2 for m, $2rs$ for n, we have $(r^2+s^2)^2\pm4rs(r^2-s^2)=\square$; an expression representing the extremes of three $\square$ numbers in arithmetical progression, $(r^2+s^2)^2$ being the mean. Multiplying the last expression by $(r^2+s^2)^2$, it becomes $(r^2+s^2)^2(r^2+s^2)^2\pm4rs(r^2-s^2)(r^2+s^2)^2=\square$.

Writing x, for $(r^2+s^2)^2$, we obtain $x^2\pm4rs(r^2-s^2)x=\square\ldots$ [***c***]. Dividing [***c***] by the square of the coefficient of x, and putting a, for the reciprocal of its coefficient, we get $a^2x^2\pm ax=\square$, and so on.

13. Make $a^2x^2+dx=\square$ . . . [1], $b^2x^2+ex=\square$ . . . [2], and $c^2x^2+fx=\square$ . . . [3] . . . . [A].

The solution of the above case of triple equality produces a formula so extensive in its application to the solution of those Diophantine Problems which require the fulfillment of six or more conditions, that we are surprised to find that no writer on Diophantine Algebra has given it a place in his work. To the student just entering upon the study of double and triple equalities, a knowledge of the extent to which this formula can be used will be found invaluable; but he will search in vain in the works of EULER, BARLOW, LAGRANGE, or any of those authors who have written on this subject, for the information he requires.

Solution. Let $a^2x^2+dx=m^2x^2$; then will $x=\dfrac{d}{m^2-a^2}$ . . . [4]. Putting this value of x in [2] and [3], and multiplying by $(m^2-a^2)^2$, they become, respectively,
$b^2d^2+(m^2-a^2)de=\square$ . . . [5], and $c^2d^2+(m^2-a^2)df=\square$ . . . [6]. These are $\square$'s if $m=\pm a$, but either of these values of m renders x infinite. If $n+a$ be substituted for m in [5] and [6], we have $b^2d^2+2aden+den^2=\square$ . . . [7], and $c^2d^2+2adfn+dfn^2=\square$ . . . [8].

Taking $bd-np$ for the root of [7], we have $n=\dfrac{2d(bp+ae)}{p^2-de}$ . . . [9].

Placing this value of n in [8], & multiplying by $(p^2-de)^2$, we get $c^2p^4+4abfp^3+(4a^2ef+4b^2df-2c^2de)p^2+4abdefp+c^2d^2e^2=\square$ . . . [10].

Assuming $cp^2-\dfrac{2abf}{c}p-cde$ for the root of [10], and reducing,

$p=\dfrac{a^2b^2f-a^2c^2e-b^2c^2d}{2abc^2}$ . . . [11]. From [4], $x=\dfrac{d}{m^2-a^2}$; or,

since $m^2=a^2+2an+n^2$, $x=\dfrac{d}{n(n+2a)}$ . . . [12].

Substituting for n in [12], its value, shown in (9), we find

$x=\dfrac{(p^2-de)^2}{4p(ap+bd)(bp+ae)}$ . . . [13]. If in [13] we substitute for p, its value, taken from [11], we shall obtain $x=$

$$\frac{\{(a^2b^2f-a^2c^2e-b^2c^2d)^2-4a^2b^2c^4de\}^2}{8a^2b^2c^2(a^2b^2f-a^2c^2e-b^2c^2d)(a^2b^2f+a^2c^2e-b^2c^2d)(a^2b^2f-a^2c^2e+b^2c^2d)}\cdots[14],$$

a, b, c, d, e, f, having any values; but if their relative values be such as to make x *negative*, it solves the triple equality

$a^2x^2-dx=\square$, $b^2x^2-ex=\square$, $c^2x^2-fx=\square$.

Therefore the foregoing work is a solution of the formulæ

$a^2x^2\pm dx=\square$, $b^2x^2\pm ex=\square$, $c^2x^2\pm fx=\square$,

the double sign being taken *disjunctively*.

Writing a for d, b for e, c for f, in [A], it becomes $a^2x^2+ax=\square$, $b^2x^2+bx=\square$, $c^2x^2+cx=\square$, and [14] is transformed to

$x=\dfrac{\{(ab-ac-bc)^2-4abc^2\}^2}{8abc(ab-ac-bc)(ab-ac+bc)(ab+ac-bc)}$ . . . [15]; the same as formula [15], p. 14.

If x be *negative*, the sign of the second term in each of the three formulæ above will be changed; therefore, when

$x=\dfrac{-\{(ab-ac-bc)^2-4abc^2\}^2}{8abc(ab-ac-bc)(ab-ac+bc)(ab+ac-bc)}$ . . . [16], (the same formula that [14], p. 14., *ought* to have been), it fulfills the conditions $a^2x^2-ax=\square$, $b^2x^2-bx=\square$, $c^2x^2-cx=\square$.

[14] solves all cases of triple equality when the terms containing x^2 have the positive sign, and each of the three formulæ consists of two terms only; one of which is some square multiple of x^2, and the other any multiple of x.

We subjoin a few of the cases, in all of which the double sign must be taken *disjunctively*, because by this formula only *one* of them can have place with the same coefficients of x^2 and x; but *which* of them is to be used depends upon the relative values of these coefficients.

1. $a^2x^2\pm ax=\square$, $b^2x^2\pm bx=\square$, $c^2x^2\pm cx=\square$.
2. $a^2x^2\pm bx=\square$, $b^2x^2\pm cx=\square$, $c^2x^2\pm ax=\square$.
3. $a^2x^2\pm cx=\square$, $b^2x^2\pm ax=\square$, $c^2x^2\pm bx=\square$.
4. $a^2x^2\pm ax=\square$, $b^2x^2\pm ax=\square$, $c^2x^2\pm ax=\square$.
5. $a^2x^2\pm(b+c)x=\square$, $b^2x^2\pm(a+c)x=\square$, $c^2x^2\pm(a+b)x=\square$.
6. $a^2x^2\pm(a+b+c)x=\square$, $b^2x^2\pm(a+b+c)x=\square$, $c^2x^2\pm(a+b+c)x=\square$.
7. $a^2x^2\pm(b-c)x=\square$, $b^2x^2\pm(a-c)x=\square$, $c^2x^2\pm(a-b)x=\square$.
8. $a^2x^2\pm(a+b-c)x=\square$, $b^2x^2\pm(b+c-a)x=\square$, $c^2x^2\pm(a+c-b)x=\square$.
9. $(a^2+b^2)x^2\pm cx=\square$, $(a^2+c^2)x^2\pm bx=\square$, $(b^2+c^2)x^2\pm ax=\square$.
10. $(a^2+b^2+c^2)x^2\pm ax=\square$, $(a^2+b^2+c^2)x^2\pm bx=\square$, $(a^2+b^2+c^2)x^2\pm cx=\square$.
11. $(a^2-b^2)x^2\pm cx=\square$, $(a^2-c^2)x^2\pm bx=\square$, $(b^2-c^2)x^2\pm ax=\square$.
12. $a^2b^2x^2\pm(a+b)x=\square$, $a^2c^2x^2\pm(a+c)x=\square$, $b^2c^2x^2\pm(b+c)x=\square$.
13. $a^2x^2\pm max=\square$, $b^2x^2\pm nbx=\square$, $c^2x^2\pm pcx=\square$.
14. $a^2x^2\pm m(a+b+c)x=\square$, $b^2x^2\pm n(a+b+c)x=\square$, $c^2x^2\pm p(a+b+c)x=\square$.
15. $a^2b^2x^2\pm(a-b)x=\square$, $a^2b^2x^2\pm(a-c)x=\square$, $a^2b^2x^2\pm(b-c)x=\square$.
16. $a^2b^2c^2x^2\pm ax=\square$, $a^2b^2c^2x^2\pm bx=\square$, $a^2b^2c^2x^2\pm cx=\square$.
17. $a^2b^2c^2x^2\pm(a+b)x=\square$, $a^2b^2c^2x^2\pm(a+c)x=\square$, $a^2b^2c^2x^2\pm(b+c)x=\square$.
18. $(a+b)^2x^2\pm ax=\square$, $(a+c)^2x^2\pm bx=\square$, $(b+c)^2x^2\pm cx=\square$.
19. $(a+b)^2x^2\pm(a+b)x=\square$, $(a+c)^2x^2\pm(a+c)x=\square$, $(b+c)^2x^2\pm(b+c)x=\square$.
20. $(a-b)^2x^2\pm(a+b)x=\square$, $(a-c)^2x^2\pm(a+c)x=\square$, $(b-c)^2x^2\pm(b+c)x=\square$; a, b, c, d, e, f, m, n, p, having any values, except in 9, 10, 11, where they must be such as to make the coefficients of x^2 $\square$'s.

Formula [15] solves No. 1, of the above cases. To solve No. 2, substitute b for d, c for e, and a for f, in [14]; then will

$$x=\frac{\{(a^3b^2-a^2c^3-b^3c^2)^2-4a^2b^3c^5\}^2}{8a^2b^2c^2(a^3b^2-a^2c^3-b^3c^2)(a^3b^2-a^2c^3+b^3c^2)(a^3b^2+a^2c^3-b^3c^2)}\cdots[17].$$

To solve No. 3, write c for d, a for e, b for f, in [14]; then

$$x=\frac{\{(a^2b^3-a^3c^2-b^2c^3)^2-4a^3b^2c^5\}^2}{8a^2b^2c^2(a^2b^3-a^3c^2-b^2c^3)(a^2b^3+a^3c^2-b^2c^3)(a^2c^3-a^3c^2+b^2c^3)}\cdots[18].$$

To solve No. 4, put a for d, e, and f, in [14], and

$$x=\frac{\{(a^3b^2-a^2c^2-b^2c^2)^2-4a^4b^2c^3\}^2}{8a^2b^2c^2(a^3b^2-a^2c^2-b^2c^2)(a^3b^2+a^2c^2-b^2c^2)(a^3b^2-a^2c^2+b^2c^2)}\cdots[19].$$

To solve No. 5, write $b+c$ for d, $a+c$ for e, and $a+b$ for f, in [14]; then we shall have

$$x=\frac{\{[a^2b^2(a+b)-a^2c^2(a+c)-b^2c^2(b+c)]^2-4a^2b^2c^4(a+c)(b+c)\}^2}{8a^2b^2c^2[a^2b^2(a+b)-a^2c^2(a+c)-b^2c^2(b+c)].[a^2b^2(a+b)-a^2c^2(a+c)+b^2c^2(b+c)].[a^2b^2(a+b)+a^2c^2(a+c)-b^2c^2(b+c)]}\cdots[20].$$

$$a^2t^2x^2+dtx=\frac{A^4}{64a^2b^4c^4v^2}+\frac{A^2d}{8a^2b^2c^2v}=\frac{A^4+8A^2b^2c^2dv}{64a^2b^4c^4v^2}=\frac{A^2}{C^2}(A^2+8b^2c^2dv),$$

$b^2t^2x^2+etx=(\frac{A}{D})^2(A^2+8a^2c^2ev)$, $c^2t^2x^2+ftx=(\frac{A}{E})^2(A^2+8a^2b^2fv)$; and since each of these three binomial factors is a $\square$,

$$\therefore\ a^2t^2x^2+dtx=(\tfrac{A}{C})^2(A^2+8b^2c^2dv)=(\tfrac{A}{C})^2(11725972997216917754494691400724)^2=\square,$$

$$b^2t^2x^2+etx=(\tfrac{A}{D})^2(A^2+8a^2c^2ev)=(\tfrac{A}{D})^2(1778601390703843690485817 32007492)^2=\square,$$

$$c^2t^2x^2+ftx=(\tfrac{A}{E})^2(A^2+8a^2b^2fv)=(\tfrac{A}{E})^2(36766587314197756737886 2842430140)^2=\square.$$

24. To find n numbers, such that if the square of the first be added to the second, the square of the second to the third, the square of the third to the fourth, ... and the square of the nth to the first, the respective sums shall all be squares.

Solution. (1) For two numbers, x, y; then $x^2+y=\square$, and $y^2+x=\square$. The first expession is a square when $y=2x+1$, and the second becomes $4x^2+5x+1=\square=\left(\frac{p}{q}x-1\right)^2$, then $x=\frac{2pq+5q^2}{p^2-4q^2}$.

(2) For three Nos., x, y, z; then $x^2+y=\square$, $y^2+z=\square$, $z^2+x=\square$. The first expression is a square when $y=2x+1$, the second when $z=4x+3$, and the third becomes $16x^2+25x+9=\square=\left(\frac{p}{q}x-3\right)^2$. Reducing, $x=\frac{6pq+25q^2}{p^2-16q^2}$. (3) For four numbers, x, y, z, w; taking $y=2x+1$, $z=4x+3$, $w=8x+7$, the first three expressions will be squares, and the fourth becomes $64x^2+113x+49=\square=\left(\frac{p}{q}x-7\right)^2$. Reducing, $x=\frac{14pq+113q^2}{p^2-64q^2}$, and thence y, z, w, are known.

(4) For n numbers, we perceive now the derivation of the successive assumptions of y, z, w, &c., from that immediately preceding, viz., multiplying the first term by 2, and the second by 2 and adding 1, as follows: $x+0$, $2x+1$, $4x+3$, $8x+7$, $16x+15$, $32x+31$, &c., in each of which assumptions, the coefficient of the first term is a power of 2, and that of the second, the same power of 2 less 1. Now, putting N=number of quantities less 1, the last assumption will be 2^Nx+2^N-1,=the value of the nth quantity, & the square of this $+x=2^{2N}x^2+(2^{2N+1}-2^{N+1}+1)x+(2^N-1)^2=\square=\left(\frac{p}{q}x-(2^N-1)\right)^2$; then will $x=\frac{2pq(2^N-1)+(2^{2N+1}-2^{N+1}+1)q^2}{p^2-2^{2N}q^2}$, and thence all the remaining quantities become known.

For three numbers, let $N=2$, then $x=\frac{6pq+25q^2}{p^2-16q^2}$, agreeing with the value of x in (2). For four Nos., let $N=3$, then $x=\frac{14pq+113q^2}{p^2-64q^2}$, agreeing with the value of x in (3), &c.; in which values of x, p, q, may be any numbers that will make the denominator positive.

By putting $N=1, 2, 3, 4$, &c., successively, each supposition will give a different value of x; thus in fact constituting answers to as many distinct problems, each perfectly general, and from which the preceding quantities, y, z, w, &c., are readily found.